NAME: Grading Key **SCIPER:** ____________

This exam is open book and open notes, but no computers are allowed.

Please answer all questions. Values of each question are given below.

Problem:	1	2	3	4	5	6	Total
Value:	25	15	15	15	15	15	100
Grade:							

Problem 1.

When there are multiple choices in the following, select all statements that are true.

1 a) Consider the linear system $x^+ = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ with the output $y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$. Which of the following statements are true?

- ○ The system is controllable
- ○ The system is observable
- ○ With the control law $u = \begin{bmatrix} 1.8 & -5.6 \end{bmatrix} x$, the closed-loop system is stable
- ○ With the control law $u = \begin{bmatrix} 1.8 & -5.6 \end{bmatrix} x$, $V(x) = x^T \begin{bmatrix} 0.9 & 0 \\ 1 & -1.6 \end{bmatrix} x$ is a Lyapunov function for the closed-loop system
- ⊗ None of the above

1 b) Consider the set $P = \cap_{i=1\ldots N} H_i$, where each H_i is a halfspace and N is finite. Which of the following statements are true?

- ○ max $a^T x$ s.t. $x \in P$ is finite for all a
- ⊗ P is a polyhedron
- ○ P is a polytope
- ⊗ P is a convex set
- ○ P may be non-convex
- ○ max $x^T x$ s.t. $x \in P$ is a convex problem

1 c) Which of the following sets are convex?

- ⊗ A slab: $\{x \in \mathbb{R}^n \mid \alpha \le a^T x \le \beta\}$
- ⊗ A hypercube: $\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \ldots, n\}$
- ⊗ The intersection $S = S_1 \cap S_2$ of two convex sets S_1 and S_2

1 d) Which of the following functions are convex?

- ◯ The quadratic function: $f(x) = x^T \begin{bmatrix} 1.5 & 0 \\ 0 & -0.2 \end{bmatrix} x$
- ⊗ The function: $f(x) = x^4$
- ⊗ The l_p norm: $f(x) = \|x\|_p$, with $p \geq 1$
- ⊗ The indicator function on a convex set $\mathbb{C}$: $f(x) = \begin{cases} 0 & x \in \mathbb{C} \\ \infty & \text{otherwise} \end{cases}$

1 e) Consider the problem $d(\lambda) = \min_x \; a \cdot x^2 + (x-1)\lambda$. Mark all correct statements

- ◯ d is convex
- ⊗ d is concave
- ◯ The convexity of d depends on a
- ◯ d is neither convex nor concave

2 f) Given the function $f(x) = \begin{cases} \infty & |x| > 1 \\ 0 & |x| \leq 1 \end{cases}$, what is $\text{prox}_{f,\rho}(v)$

- ◯ $\text{prox}_{f,\rho}(v) = 0$
- ◯ $\text{prox}_{f,\rho}(v) = \infty$
- ◯ $\text{prox}_{f,\rho}(v) = v$
- ⊗ $\text{prox}_{f,\rho}(v) = \begin{cases} 1 & v \geq 1 \\ -1 & v \leq -1 \\ v & \text{otherwise} \end{cases}$
- ◯ $\text{prox}_{f,\rho}(v) = \begin{cases} 1 & 1 \leq v \leq 1 + \sqrt{2/\rho} \\ -1 & -1 - \sqrt{2/\rho} \leq v \leq -1 \\ v & \text{otherwise} \end{cases}$

1 g) The maximum control invariant set of a linear system subject to polyhedral constraints is

- ⊗ Convex
- ◯ Polytopic
- ◯ None of the above

1 h) Let S be an invariant set for the linear system $x^+ = Ax$. For which values of α is αS also an invariant set for this system?

- [x] $\alpha = 0$
- [x] $0 < \alpha < 1$
- [x] $\alpha > 1$
- [x] $\alpha = 1$

2 i) Let $x^+ = f(x, u)$ be a system with constraints $(x, u) \in X \times U$, and let C be the maximal control invariant set of the system. Mark the true statements.

- [x] There exists an $x \in X$ and a $u \in U$ such that $f(x, u) \in C$
- [ ] For all $x \in X$, there exists a $u \in U$ such that $f(x, u) \in C$
- [ ] There exists an $x \in X \backslash C$ and a $u \in U$ such that $f(x, u) \in C$
- [x] There does not exist an $x \in X \backslash C$, and a $u \in U$ such that $f(x, u) \in C$

2 j) Let $x^+ = f(x, u)$ be a system with constraints $(x, u) \in X \times U$, and let X_∞ be the maximal invariant set of the system $x^+ = f(x, \kappa(x))$ for some controller $\kappa(x)$. Mark the true statements.

- [ ] For all $x \in X$, there exists a $u \in U$ such that $f(x, u) \in X_\infty$
- [x] For all $x \in X_\infty$, there exists a $u \in U$ such that $f(x, u) \in X_\infty$
- [x] It is possible that there exists an $x \in X \backslash X_\infty$ and $u \in U$ such that $f(x, u) \in X_\infty$
- [ ] For all $x \in X \backslash X_\infty$, there doesn't exist any $u \in U$ such that $f(x, u) \in X_\infty$

2 k) If C is a control invariant set for the system $x^+ = f(x, u)$, then C is a positively invariant set for the autonomous system $x^+ = f(x, \kappa(x))$, under which of the following control laws

- [ ] $\kappa(x) = 0$
- [ ] $\kappa(x) = Kx$, where K is a stabilizing control gain
- [ ] $\kappa(x) = \text{argmin}\,\{\|u\|_2\}$
- [x] $\kappa(x) = \text{argmin}\,\{\|u\|_2 \mid f(x, u) \in C\}$
- [ ] all of the above
- [ ] none of the above

1 l) A linear autonomous system with convex constraints has two invariant sets S_1 and S_2. Indicate which of the following statements hold

- [ ] $S_1 \cap S_2$ is an invariant set of the system
- [x] $S_1 \cup S_2$ is an invariant set of the system
- [x] The convex hull of $S_1 \cup S_2$ is an invariant set of the system

1 m) Given a linear system $x^+ = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ and a state-feedback control law $u = \begin{bmatrix} -3 & 1 \end{bmatrix} x$, which of the following is a Lyapunov function for the closed-loop system $x^+ = (A + BK)x$?

- ○ $x^T x$
- ○ $\frac{\|x+3\|_\infty}{x^T x}$
- ○ $x^T \begin{bmatrix} 1 & 0.7 \\ -0.3 & 0.9 \end{bmatrix} x$
- ⊗ None of the above

2 n) Consider the following four MPC problems.

$$J_1(x) = \min \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i \quad \text{s.t.} \quad x_{i+1} = Ax_i + Bu_i, \quad (x_i, u_i) \in X \times U, \quad x_0 = x$$

$$J_2(x) = \min \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + J_1(x_N) \quad \text{s.t.} \quad x_{i+1} = Ax_i + Bu_i, \quad (x_i, u_i) \in X \times U, \quad x_0 = x$$

$$J_3(x) = \min \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i \quad \text{s.t.} \quad x_{i+1} = Ax_i + Bu_i, \quad x_0 = x$$

$$J_4(x) = \min \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + J_3(x_N) \quad \text{s.t.} \quad x_{i+1} = Ax_i + Bu_i, \quad (x_i, u_i) \in X \times U, \quad x_N \in X_f, \quad x_0 = x$$

with $Q \succ 0$, $R \succ 0$ and $X_f \subseteq X$ an invariant set for the system $x^+ = Ax + B\kappa(x)$, where $\kappa(x)$ is the control law defined by the MPC problem on the bottom left. Mark all correct statements.

- ⊗ $J_1(x) \leq J_2(x)$
- ⊗ $J_1(x) \geq J_2(x)$
- ⊗ $J_1(x) = J_2(x)$
- ⊗ $J_3(x) \leq J_1(x)$
- ○ $J_3(x) \geq J_1(x)$
- ○ $J_3(x) = J_1(x)$
- ⊗ $J_3(x) \leq J_4(x)$
- ○ $J_3(x) \geq J_4(x)$
- ○ $J_3(x) = J_4(x)$

1

o) Consider the uncertain linear system $x^+ = Ax + Bu + w$ and two disturbance sets W_1 and W_2 such that $W_1 \subset W_2$. Which of the following statements is true for a polytopic set Ω

- ☒ $pre^{W_1}(\Omega) \supseteq pre^{W_2}(\Omega)$
- ○ $pre^{W_1}(\Omega) \subseteq pre^{W_2}(\Omega)$
- ○ Nothing can be said without knowledge of the matrices A and B

1

p) Consider a linear system $x^+ = Ax + Bu$ and the MPC controller

$$
\begin{aligned}
J^\star(x) = \min \ & \sum_{i=0}^{N-1} l(x_i, u_i) \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
& (x_i, u_i) \in (X \times U) \\
& x_0 = x
\end{aligned}
$$

When running the controller, we observe that the system is subject to a lot of noise, and that the optimization problem is sometimes infeasible. Which of the following methods will ensure recursive feasibility?

- ○ Use a longer horizon
- ○ Add a terminal constraint $x_N \in X_f$
- ○ Add a terminal cost $V_f(x_N)$
- ○ Use a tracking MPC formulation
- ☒ Use a soft-constrained MPC formulation

2

q) Consider the following MPC controller

$$
\begin{aligned}
J^\star(x) = \min \ & \sum_{i=0}^{N-1} q^T x_i + r^T u_i + p^T x_N \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
& (x_i, u_i) \in (X \times U) \\
& x_N = 0 \\
& x_0 = x
\end{aligned}
$$

where $0 \in U$.

- ☒ The MPC controller is recursively feasible
- ○ The MPC controller will asymptotically stabilize the system $x^+ = Ax + Bu$
- ○ Neither of the above

2 r) Consider the standard (top) and soft-constrained (bottom) MPC problem formulations

Problem 1

$$
\begin{aligned}
J^\star(x) \quad = \min_{u,x} \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& G x_i \le g \\
& G_N x_N \le g_N \\
& H u_i \le h \\
& x_0 = x
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
J^\star_{soft}(x) \quad = \min_{u,x,\epsilon} \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T P x_N + \rho \sum_{i=0}^{N} \epsilon_i^T \epsilon_i \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i \\
& G x_i \le g + \epsilon_i \\
& G_N x_N \le g_N + \epsilon_N \\
& H u_i \le h \\
& x_0 = x \\
& \epsilon_i \ge 0
\end{aligned}
$$

Mark all the correct statements

- ☒ $J^\star_{soft}(x) \le J^\star(x)$ for all x feasible for both problems
- ○ $J^\star_{soft}(x) \ge J^\star(x)$ for all x feasible for both problems
- ○ $J^\star_{soft}(x) \ge J^\star(x)$ if ρ is sufficiently large
- ☒ For a given x, Problem 1 is feasible if Problem 2 is feasible
- ☒ For a given x, Problem 2 is feasible if Problem 1 is feasible

Problem 2.

/15

Consider the linear system $x^+ = Ax$ subject to the constraints $x \in X$. Let $Y \subseteq X$ and $Z \subseteq X$ be invariant sets for this system and $x_s \in \operatorname{int} X$[1] a steady-state solution, $x_s = Ax_s$.

1. Prove that the set $\bar{Y} = Y \oplus \{x_s\} = \{x + x_s \,|\, x \in Y\}$ is also an invariant set for this system if $\bar{Y} \subseteq X$

5

Let $\bar{x}$ be in $\bar{Y}$. There exists an $x \in Y$ such that $\bar{x} = x + x_s$. Consider the successor state $A\bar{x} = Ax + Ax_s = Ax + x_s$. By the invariance of Y, we have that $Ax \in Y$ and therefore $A\bar{x} = Ax + x_s$ is in $\bar{Y}$.

5

2. Prove that there exists a scaling factor $\lambda > 0$ such that $\{x_s\} \oplus \lambda Z := \{\lambda x + x_s \,|\, x \in Z\}$ is an invariant set for the system (Note that $\{x_s\} \oplus Z$ may, or may not be a subset of X)

We need to show that for each x_s there exists a $\lambda > 0$ such that $\{x_s\} \oplus \lambda Z \subseteq X$, and then we can apply the result from the first question. Such a λ always exists because x_s is in the interior of X, and not on the boundary.

[1]Note : $\operatorname{int} X := \{x \,|\, \exists \epsilon > 0 \,, x + y \in X, \, \forall y \in \mathbb{B}_\epsilon\}$ refers to the interior of X, where $\mathbb{B}_\epsilon := \{y \,|\, \|y\| \leq \epsilon\}$

5

3. Let $A = 1$, $X = [-2, 2]$ and $Z = [-1, 1]$. Compute the largest λ as a function of the steady-state x_s

All values $-2 < x_s < 2$ are valid steady-states.

$$\lambda_{max} = 2 - |x_s|$$

Problem 3.

/15

Consider the linear discrete time system

$$x^+ = 1.1x + u + w$$

where the state x, the input u and the disturbance w, are all of dimension one and are constrained as follows

$$X = [-2, 2] \qquad U = [-0.8, 0.8] \qquad W = [-0.2, 0.1]$$

Your goal is to design a tube-based MPC controller for this system.

6

1. Compute the minimal robust invariant set $\mathcal{E}$ for this system with the control law $u = Kx$, for $K = -0.3$

   Hint : $[a, b] \oplus [c, d] = [a + c, b + d]$

2 for alg

Autonomous system is $x^+ = (A + BK)x = 0.8x$

$$\begin{aligned}
\Omega_0 &= [-0.2, 0.1] \\
\Omega_1 &= 0.8 \cdot [-0.2, 0.1] + [-0.2, 0.1] \\
\Omega_2 &= 0.8^2 \cdot [-0.2, 0.1] + 0.8 \cdot [-0.2, 0.1] + [-0.2, 0.1] \\
&\vdots \\
\mathcal{E} &= [-0.2 \cdot \sum 0.8^i, 0.1 \cdot \sum 0.8^i] \\
&= [-0.2 \cdot 5, 0.1 \cdot 5] \\
&= [-1, 0.5]
\end{aligned}$$

4

2. Compute the tightened constraints $\tilde{X} = X \ominus \mathcal{E}$ and $\tilde{U} = U \ominus K\mathcal{E}$

2 $\tilde{X} = [-2, 2] \ominus [-1, 0.5] = [-1, 1.5]$

2 $\tilde{U} = [-0.8, 0.8] \ominus -0.3 \cdot [-1, 0.5] = [-0.8, 0.8] \ominus [-0.15, 0.3] = [-0.65, 0.5]$

5

3. Verify that the set $\mathcal{X}_f = [-1, 1.5]$ if used as a terminal constraint for the tube-based MPC scheme will result in a recursively feasible controller for the terminal control law $u = -0.3x$

The set X_f must be invariant for the nominal system subject to the tightened constraints.
Invariance:

$$(A + BK)X_f = 0.8 \cdot [-1, 1.5] = [-0.8, 1.2] \subset X_f \qquad \checkmark$$

Constraint satisfaction:

$$X_f \subseteq X \ominus \mathcal{E} \Leftrightarrow [-1, 1.5] \subseteq [-1, 1.5] \qquad \checkmark$$
$$KX_f \subseteq U \ominus K\mathcal{E} \Leftrightarrow -0.3 \cdot [-1, 1.5] \subseteq [-0.65, 0.5]$$
$$\Leftrightarrow [-0.45, 0.3] \subseteq [-0.65, 0.5] \qquad \checkmark$$

Problem 4. /15

Consider the following parametric QP problem with the parameter x.

$$
\begin{aligned}
f^\star(x) = \min_z\ & z^2 + (x+1)z + x \\
\text{s.t.}\ & z \geq 2x \\
& z \geq 0
\end{aligned}
$$

4

1. Give matrices M, Q and vector q such that the optimal solution of the problem above is a linear transformation of the solution $y(x)$ to the following parametric LCP

$$w - My = Qx + q \qquad w, y \geq 0 \qquad w^T y = 0 \qquad (1)$$

$2z + x + 1 - \lambda - v = 0$

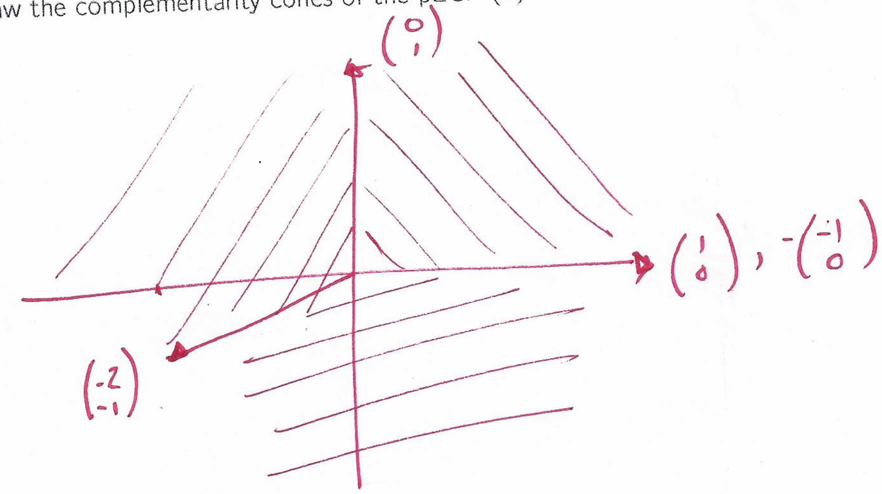
$-s + z = 2x \quad , z, s \geq 0$

$\lambda, v \geq 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} v \\ s \end{pmatrix} - \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}_{M}\begin{pmatrix} z \\ \lambda \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{Q} x + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{q}$$

4

2. Draw the complementarity cones of the pLCP (1)



7

3. Compute the optimal value function $f^*(x)$

Write the answer here:

$$f^*(x) = \begin{cases} (-x^2+1)/4 & x \in [-\infty, -1] \\ x+1 & x \in [-1, 0] \\ 6x^2+3x & x \in [0, \infty] \end{cases}$$

1) $v=0,\ s=0$

$$\begin{pmatrix} z \\ \lambda \end{pmatrix} = -\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^{-1}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = +\begin{bmatrix} 0 & -1 \\ +1 & -2 \end{bmatrix}\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} +2 \\ +5 \end{pmatrix}x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq 0$$

$\boxed{x \geq 0}$, $x \geq -1/5$

$$f^*(x) = (+2x)^2 + (x-1)(+2x) + x$$
$$= 4x^2 + 2x^2 + 2x + x$$
$$= 6x^2 + 3x$$

2) $v=0,\ \lambda=0$

$$\begin{pmatrix} z \\ s \end{pmatrix} = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix}^{-1}\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{-2}\cdot\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}\left[\begin{pmatrix} 1 \\ -2 \end{pmatrix}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$= -\frac{1}{2}\cdot\left[\begin{bmatrix} 1 \\ +5 \end{bmatrix}x + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} -1/2 \\ -5/2 \end{pmatrix}x + \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix} \geq 0$$

$\boxed{x \leq -1}$, $5x \leq -1 \Rightarrow x \leq -1/5$

$$f^*(x) = \left(-\frac{1}{2}x - \frac{1}{2}\right)^2 + (x-1)\left(-\frac{1}{2}x - \frac{1}{2}\right) + \left(-\frac{1}{2}x - \frac{1}{2}\right)$$
$$= \frac{1}{4}\left[x^2 + 2x + 1\right] - \frac{1}{2}\cdot\left[x^2 + \cancel{x} - \cancel{x} - 1\right] - \frac{1}{2}\cdot\left[x+1\right]$$
$$= \frac{x^2}{4} + \cancel{\frac{x}{2}} + \frac{1}{4} - \frac{x^2}{2} + \frac{1}{2} - \cancel{\frac{x}{2}} - \frac{1}{2}$$
$$= -\frac{x^2}{4} + \frac{1}{4}$$

3) $z, \lambda = 0$

$$\begin{pmatrix} v \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \geq 0 \Rightarrow \boxed{\begin{matrix} x \geq -1 \\ x \leq 0 \end{matrix}}$$

$$f^*(x) = x + 1$$

Problem 5. /15

Consider the linear system $x^+ = Ax + Bu$ and the following unconstrained MPC problem

$$\begin{aligned}
\min \quad & \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + x_N^T S x_N \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i \\
& x_0 = x
\end{aligned}$$

1. Assume that $\rho(A) < 1$, $Q \succ 0$ and $R \succ 0$.
Give a sufficient condition on the matrix S that ensures stability of the closed-loop system with the terminal control law $u = 0$.

> We require that the terminal weight must be a Lyapunov function in the terminal set for the terminal controller. The controller is $u = 0$, and so the terminal set is $\mathbb{R}^n$. The Lyapunov condition is
>
> $$x^T A^T S A x - x^T S x \le -x^T Q x$$
> $$\Leftrightarrow \quad A^T S A - S \le -Q$$

2. Assume that $\rho(A) > 1$, $Q \succ 0$ and $R \succ 0$.
Give a condition on the matrix S such that the resulting MPC control law is equal to the infinite-horizon LQR solution. (You can use any terminal control law you like)

> We require that S is the solution to the discrete algebraic Riccati equation
>
> $$S = Q + A^T S A - A^T S B (R + B^T S B)^{-1} B^T S A$$
>
> The optimality principle will then ensure that the finite-horizon solution is equal to the infinite.

3. Assume that $Q = 0$, $R = 1$ and $B = 1$.

5 i) Use dynamic programming to compute the MPC control law for $A = -0.5$, $S = -0.5$ and $N = 2$.

$V_2(x) = -0.5 \cdot x^2$

$V_1(x) = \min_u \; u^2 + -0.5 \cdot (-0.5x + u)^2 = -x^2/4$

$\cancel{V_0(x) = \min_u \; u^2 - 1/4 \cdot (-0.5x + u)^2 = -1/12x^2}$

$\cancel{K(x) = \text{argmin}_u \, u^2 - 1/12 \cdot (-0.5x + u)^2 = -x/22}$

Closed-loop system is $\cancel{x^+ = (-1/2 - 1/22)x = -6/11}x$. Stable.

$$K(x) = \text{argmin}_u \; u^2 - \frac{1}{4} \cdot (-0.5x + u)^2$$

$$\nabla = 0 = 4u + \frac{1}{2}\left(\frac{x}{2} - u\right)$$

$$-\frac{x}{2} = 3u \Rightarrow u = -x/6$$

2 ii) Is the resulting closed-loop system stable, or unstable?

Stable. Eigenvalues in the unit ball.

$$x^+ = \frac{-3x}{6} - \frac{x}{6} = -\frac{4}{6} \cdot x$$

/15

Problem 6.

Consider the convex optimization problem

$$\begin{aligned} &\min\ x^TQx + q^Tx + r \\ &\text{s.t. } \|Px - p\|_2^2 \le 1 \end{aligned} \tag{2}$$

1. Derive a closed-form expression for the proximal operator of the function

4

$$f(y) = \begin{cases} 0 & \|y\|_2^2 \le 1 \\ \infty & \text{otherwise} \end{cases}$$

Hint : Recall the definition of the proximal operator

$$\text{prox}_{f,\rho}(v) := \text{argmin}_y\ f(y) + \frac{\rho}{2}\|y - v\|_2^2$$

$$\begin{aligned} \text{prox}_{f,\rho}(v) &= \text{argmin}_y\ \|y - v\|_2^2 \\ &\qquad \text{s.t. } y^Ty \le 1 \\ &= \begin{cases} v & \|v\| \le 1 \\ \frac{v}{\|v\|} & \text{otherwise} \end{cases} \end{aligned}$$

2. Give functions f, g and matrices A, B and b so that problem (3) is equivalent to (2).

4

$$\begin{aligned} &\min\ f(x) + g(y) \\ &\text{s.t. } Ax + By = b \end{aligned} \tag{3}$$

$f(x) = x^TQx + q^Tx + r$

$g(y) =$ indicator function for 2-norm unit ball

$A = P$

$B = -I$

$b = p$

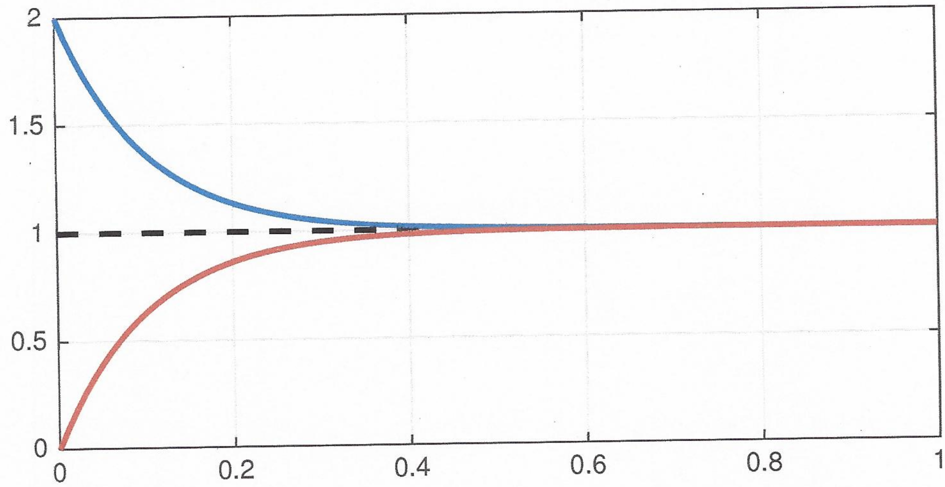
3. Give the three steps of the ADMM algorithm for this problem, using the prox operator that you derived in Part 1.

4

$$\begin{aligned}
x^{k+1} &= \text{argmin}_x\ f(x) + \frac{\rho}{2}\|Ax + By^k - b + \mu^k\|^2 \\
&= \text{argmin}_x\ x^TQx + q^Tx + r + \frac{\rho}{2}\|Px - y^k - p + \mu^k\|^2 \\
&= -(2Q + \rho P^TP)^{-1}(q + \rho P^T(-y^k - p + \mu^k)) \\
y^{k+1} &= \text{argmin}_y\ g(y) + \frac{\rho}{2}\|Ax^{k+1} + By - b + \mu^k\|^2 \\
&= \text{prox}_{g,\rho}(Px^{k+1} - b + \mu^k) \\
\mu^{k+1} &= \mu^k + Ax^{k+1} + By^{k+1} - b \\
&= \mu^k + Px^{k+1} - y^{k+1} - p
\end{aligned}$$

4. Suppose that we solve problem (2) using (a) A logarithmic barrier method and (b) ADMM. The figure below shows the value of the objective function during the optimization for each case, with the optimal value marked by a dashed line.

3



The upper line must be the barrier method, and the lower ADMM because the barrier method takes feasible iterates and therefore the function value during optimization must upper bound the optimal solution.

Which line corresponds to ADMM? Explain your answer.